Model theory - 2^{nd} lecture - Lowenheim - Skolem theorem

Let Φ be a theory and $\mathcal{M}$ be a model, when a formula φ is true in $\mathcal{M}$ we say "$\mathcal{M}$ satisfies the formula φ", in symbols $\mathcal{M} \models \varphi$

$\vdash$ (logical) entailment $\qquad$ $\models$ semantic entailment

$\Phi \vdash \varphi$ means φ is a consequence of Φ from a proof theoretic level

$\Phi \models \varphi$ means every model of Φ satisfies φ

Def a theory is "satisfiable" if it has a model

Def a theory Φ is "deductively closed" if every theorem of Φ is an axiom, i.e. $\Phi \vdash \varphi$ iff $\varphi \in \Phi$

Def a theory Φ is "complete" if, for every formula φ, either

$\Phi \models \varphi$ or $\Phi \models \neg\varphi$ $\qquad$ (For example groups have a theory which is <u>not</u>' complete (abelianity))

How to present consistent theories?

Take Φ consistent and satisfiable, $\mathcal{M} \models \Phi$

Now you define $\Phi_{\mathcal{M}} := \{\varphi \mid \mathcal{M} \models \varphi\}$

Question Can you name another complete theory?

Theorem (FOL Completeness) $\Phi \vdash \varphi$ iff $\Phi \vDash \varphi$

Theorem (FOL Compactness) Φ is satisfiable iff every finite subset of Φ has a model

Lowenheim - Skolem theorems

In the days of these theorems FOL was considered the whole of logic

Q Does the theory of Peano charachterize natural numbers?

I e $M \models PA$ then $M \cong \mathbb{N}$? (No, but not easy)

the structure are isomorphic

$Mod(\varphi)$ is almost always a proper class

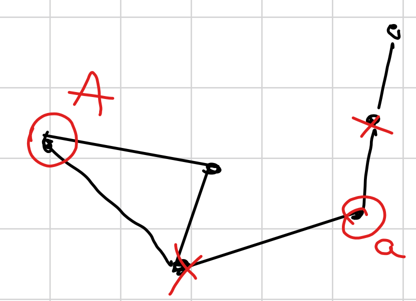
The following proofs are Tarski's

one can do with other cardinalities *

Theorem (LS downwards) Let φ be a theory in a finite language.
Let M be an infinite model of φ. Let A be a finite subset of M
Then, there exists a countable model of φ between A and M

Example φ is the theory of graphs where each node has two distinct nodes to which it's related

φ $\quad \forall x \quad \exists y, z \ ((y \neq z) \wedge (x R y) \wedge (x R z))$



Notice how $\boxed{\exists x \quad x R a} \equiv \varphi^a(x)$ $\quad G \models \varphi^a(x)$ but A doesn't

The idea is to make a chain $A \subset B_1 \subset B_2 \subset \ \ldots \subset B_n \subset \ \ldots \subset M$

Then $\bigcup_{n \in \mathbb{N}} B_n$ will be a model of φ I build B_{n+1}

so that for every formula $\varphi(x) \equiv \exists x \ x R b$ that is modelled by M

I choose a $\bar{b}$ in G such that $\varphi(\bar{b})$ is true $B_{n+1} = B_n \cup \{b_\varphi\}$

Proof By induction on the complexity of the formula (the only difficult step is $\varphi(x) = \exists x \, \psi(x, \bar{a})$) ■

Theorem (LS upwards) Let T be a theory in a finite language $\mathcal{L}$. Let M be a countable model of T. Then, for every cardinal $\lambda \geq \aleph_0$, there exists a model N of cardinality λ s.t. $M \subseteq N$

Proof M models T. I define a new language $\mathcal{L}_\lambda = \mathcal{L} \cup \{c_i\}_{i \in \lambda}$ where every c_i is a constant, and define $T_\lambda = T \cup \{c_i \neq c_j\}$.
Compactness guarantees T_λ has a model, it will be a model for T and it will be at least of cardinality λ ■